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Module 10: Mixed model theory II

Tests and confidence intervals

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Aim of this module

- Continue from the first theory module
- Have seen: Definition and estimation
- Missing: Tests and confidence intervals
- How are tests computed (fixed and variance parameters)
- How are confidence intervals constructed (fixed and variance parameters)

Summary of first theory module

- Any mixed model can be expressed as:

$$\mathbf{y} \sim N(\mathbf{X}\beta, \mathbf{V}(\gamma))$$

Where $\mathbf{V}(\gamma) = \mathbf{ZGZ}' + \mathbf{R}$

- The parameters of the model are estimated by minimizing:

$$\ell_{re}(\mathbf{y}, \beta, \gamma) \propto \frac{1}{2}\left\{\log|\mathbf{V}(\gamma)| + (\mathbf{y}-\mathbf{X}\beta)'(\mathbf{V}(\gamma))^{-1}(\mathbf{y}-\mathbf{X}\beta) + \log|\mathbf{X}'(\mathbf{V}(\gamma))^{-1}\mathbf{X}|\right\}$$

Formulating a hypothesis

- A linear hypothesis (what we normally want) is formulated as:

$$\mathbf{L}'\beta = c$$

- Two examples both for the simple one way ANOVA model with three treatments:

$$y_i = \mu + \alpha(\texttt{treatment}_i) + \varepsilon_i$$

1) Same as $\alpha_1 - \alpha_2 = 0$

$$\underbrace{\begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}}_{\mathbf{L}'} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

2) Same as $\alpha_1 = \alpha_2 = \alpha_3$, or the usual test for removing treatment from the model

$$\underbrace{\begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}}_{\mathbf{L}'} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \mathbf{0}$$

Testing fixed effects

- The estimate of $\mathbf{L}'\beta$ is $\mathbf{L}'\widehat{\beta}$.
- From first theory module we know $\widehat{\beta}$, so:

$$\mathbf{L}'\widehat{\beta} = \mathbf{L}'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$$

- A few matrix calculations give:

$$\mathbf{L}'\widehat{\beta} \sim N(\mathbf{L}'\beta, \mathbf{L}'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{L})$$

- So if the hypothesis $\mathbf{L}'\beta = c$ is true we have:

$$(\mathbf{L}'\widehat{\beta} - c) \sim N(0, \mathbf{L}'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{L})$$

- The so called Wald test becomes

$$W = (\mathbf{L}'\widehat{\beta} - c)'(\mathbf{L}'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{L})^{-1}(\mathbf{L}'\widehat{\beta} - c)'$$

  W is approximately $\chi^2_{\text{rank(L)}}$–distributed

Better approximation: Wald F–test & Satterthwaite's

- Wald F–test:

$$F = \frac{W}{df_1}$$

- Assumed asymptotically F–distributed
- df_1 is the number of parameters "eliminated" by the hypothesis (rank($\mathbf{L}$))
- df_2 is estimated to make the F–distribution fit (see later)
- The P–value is computed by:

$$P_{\mathbf{L}'\boldsymbol{\beta}=c} = P(F_{df_1,df_2} \geq F)$$

- These tests are in proc mixed ANOVA table if `/ddfm=satterth` is specified

Confidence intervals of fixed effects

- For a one dimensional linear combination $\mathbf{L}'\beta$
  (For instance $\alpha_1 - \alpha_2$)
- We know the estimate $\mathbf{L}'\widehat{\beta}$
- We know the standard deviation $\sqrt{\mathbf{L}'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{L}}$
- So the 95% confidence interval based on the t–distribution becomes

$$\mathbf{L}'\widehat{\beta} \pm t_{0.975,df}\sqrt{\mathbf{L}'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{L}}$$

- Satterthwaite's is used to determine df

The `estimate` and the `contrast` statements

- These linear combinations $\mathbf{L}'\beta$ can be specified directly in SAS
- `estimate` does the one dimensional with confidence interval
- `contrast` comparing several treatments (rank($\mathbf{L}$) > 1)
- Example: One way ANOVA with five treatments and random blocks

$$\beta' = (\mu \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5)$$

```
proc mixed;
  class treatment block;
  model y = treatment/ddfm=satterth;
  random block;
  estimate 'tmt1-tmt2' treatment 1 -1 0 0 0/cl;    α1 − α2
  contrast 'tmt1=tmt2=tmt3' treatment 1 -1  0  0  0,    α1 − α2
                            treatment 1  0 -1  0  0;    α2 − α3
run;
```

Test of random effects parameters

- The likelihood ratio test can be applied to test $A \to B$ if B is a sub–model of A
- A the model including some variance parameter
- B the model without that variance parameter
- The test is calculated as:

$$G_{A\to B} = 2\ell_{re}^{(B)} - 2\ell_{re}^{(A)}$$

  (These are directly in the SAS output)

- Asymptotically $G_{A\to B}$ is χ_1^2–distributed
- So the P–value is:

$$P(\chi_1^2 \geq G_{A\to B})$$

Confidence intervals for random effects parameters

- We start by assuming that asymptotically: $\widehat{\sigma}_b^2 \sim \frac{\sigma_b^2}{df}\chi_{df}^2$
- The 95% confidence interval takes the well known form:

$$\frac{df\widehat{\sigma}_b^2}{\chi^2_{0.025;df}} < \sigma_b^2 < \frac{df\widehat{\sigma}_b^2}{\chi^2_{0.975;df}}\ ,\quad \text{but we still don't know } df$$

- The (theoretical) variance is:

$$\text{var}\left(\frac{\sigma_b^2}{df}\chi_{df}^2\right) = \frac{2\sigma_b^4}{df}$$

- From the curvature of the likelihood we can estimate the actual variance of our estimate
- Matching these two things:

$$\text{var}(\widehat{\sigma}_b^2) = \frac{2\sigma_b^4}{df}$$

- Solving for df gives Satterthwaite's approximation in this case:

$$\widehat{df} = \frac{2\widehat{\sigma}_b^4}{\text{var}(\widehat{\sigma}_b^2)}$$

- This is omputed by SAS if we write something like:

```
proc mixed cl;
  class treatment block;
  model y = treatment/ddfm=satterth;
  random block;
run;
```